Bohmian mechanics

Kvantmekanik F3

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31st January 2008

Contents

Part I
Bohmian mechanics

1 Introduction

In the 1928 Solvay congress [22] Louis de Broglie presented an equation of motion that could account for the quantum interference phenomena observed which couldn't be explained in the framework of classical physics. This interpretation of quantum mechanics was soon to be abandoned for the Copenhagen interpretation, but in 1952 David Bohm [11] published an article in which he presented an interpretation of quantum theory similar to that of de Broglie, later to be known as Bohmian mechanics or de Broglie-Bohm theory, and showed that this interpretation reproduced the results of ordinary quantum mechanics. According to Bohmian mechanics, particles have a distinct position and move according to a universal pilot wave. Distinguishing features of Bohmian mechanics as compared to ordinary quantum mechanics is that it is deterministic and manifestly non-local. This theory has philosophical implications desirable to many, most notably its determinism and the fact that it gives particles exact positions, as opposed to the probabilistic wavefunction interpretation of ordinary quantum mechanics.

Other alternative deterministic interpertations exist. Notably Gerardus 't Hooft, Nobel prize lauerate in physics of 1999, has presented the idea that underlying quantum mechanics is a more fundamental Planck scale theory [27]. His idea is that quantum mechanics emerges as a statistical abstraction from a complex sub-quantum world through the process of *information loss*. The proposition of 't Hooft borrows many concepts from information theory since

> "[at] that scale, all we have is *bits of information*." [27] (emphasis in original)

In the first part of this report we will describe the history and theory of Bohmian mechanics, and discuss its non-locality and implications for the interpretation of the measurement problem. The second part will deal with simulations of Bohmian mechanics.

2 History

The first notion of something similar to the pilot wave of Bohmian mechanics was introduced by Albert Einstein, who initially thought that guidance by the electromagnetic field might explain interference phenomena of particle-like photons [18]. It wasn't long until he bumped into problems with this theory though, and he soon abandoned it. At the Fifth Solvay International Conference, on *Electrons and Photons*, the newly formulated quantum theory was discussed, following the discovery of Schrödinger's equation the previous year [22]. Louis de Broglie was invited to give a lecture on wave mechanics and its interpretation, and described how the equation of particle motion, later to be known as the guiding equation in Bohmian mechanics, could explain quantum interference. He didn't manage to convince the illustrious audience, containing 17 Nobel Prize laureates, specifically failing to respond satisfactorily to an objection concerning inelastic scattering presented by Wolfgang Pauli. Pauli's argument was that, according to de Broglie's theory, the particle momenta fluctuate and never reach a stationary energy following a particle's inelastic scattering by a hydrogen atom. Since this is known to be wrong from experiments, de Broglie's interpretation couldn't be correct. The problem with this argument, as stated by Bohm [11], was that the incoming waves were assumed to be of infinite extent, a case which is not realisable in practice.

As the Copenhagen interpretation harvested success with its elegant and effective mathematics, while the pilot wave theory struggled with mathematical problems, de Broglie abandoned the latter. Another big setback for the Bohmian mechanics came in 1932, when von Neumann presented a proof against dispersion free states, which thus ruled out all hidden variables theories [32]. His own conclusion was that

> "[it] is therefore not, as is often assumed, a question of reinterpretation of quantum mechanics - the present system of quantum mechanics would have to be objectively false in order that another description of the elementary process than the statistical one be possible."

Thus, Bohmian mechanics could finally be rejected.

In 1951, however, David Bohm rediscovered the pilot wave theory and started to work on it again. In 1952 he presented an article in which he showed that all predictions from quantum theory were reproducible by the pilot wave theory [11]. His hopes were that it would be possible to handle the problems that ordinary quantum theory meet at subatomic length scales with only small modifications in the Bohmian interpretation, but these hopes where later abandoned.

The theory seemed to be consistent, and thus one had a theory that was proven impossible by von Neumann. A lot of effort was put into examining von Neumann's proof, searching for ways to verify that it was applicable also to Bohmian mechanics. In 1966 it was found that one of the assumptions made by von Neumann ruled out hidden variables theories such as de Broglie theory from the very beginning. This was supposedly discovered by Grete Hermann already in 1935, but wasn't publicly acknowledged until Bell rediscovered her findings [3, 19]. In the same article Bell also refutes the impossibility proofs by Jauch-Piron and by Gleason [3].

Bohm's paper also encouraged de Broglie to return to Bohmian mechanics, but most of the work in this field during the next decades was carried out by Bohm and Bell.

Bell worked for several years at CERN, designing accelerators, but also put a lot of thought and work into his articles on the foundations of quantum mechanics. His most famous result is the famous theorem which bears his name. *Bell's Theorem* [2] has often been presented as the definitive argument against hidden variable interpretations of quantum mechanics, but this was not a position held by Bell himself. It does, however, rule out theories relying on locality as proposed by Einstein [17]. Throughout his career, he examined the "orthodox" interpretation in numerous articles, the most important of which are collected in [10]. His search for a "precise" and "serious" formulation of quantum mechanics [8] leaves its mark on all of those articles, from his first refutation in 1966 of the numerous "proofs" of the impossibility of hidden variables [3], to the book's final article from 1990 [9]. The latter deals explicitly with the local causality issues at the heart of Bell's theorem, their implications for quantum mechanics in general and specifically for hidden variable theories. Though the interpretation of quantum mechanics has lead to many a heated debate, his rhetoric always retains a pleasant tone. Especially his later works are marked not only by a keen scientific eye, but also by a sharp and sardonic sense of humour.

> "Was the wavefunction of the world waiting to jump for thousands of years until a single-celled living creature appeared? Or did it have to wait for a little longer, for some better qualified system... with a Ph.D.?" [8]

An important contribution to Bohmian mechanics was Bell's work on clarifying the axioms of the theory [15], but he also did work on spin and quantum field theory [5].

The more active scientists within Bohmian mechanics nowadays comprise for example Sheldon Goldstein, Detlef Dürr, Roderich Tumulka, Nino Zanghí and Antony Valentini. A lot of the work has been focused on reproducing the results of quantum field theory and relativistic generalizations of the theory [23].

3 Theory

The Schrödinger equation for a system of N particles in three dimensions is written [1]

$$\mathrm{i}\hbar\dot{\psi}(\boldsymbol{q},t) = \hat{\mathbf{H}}\psi(\boldsymbol{q},t). \tag{1}$$

Here the $\boldsymbol{q} = [x_1, \ldots, x_{3N}]$ denotes the $3N$ spatial variables of the N particles, while t is the time variable. If $\boldsymbol{x}_i$ is the position of particle i, the equation of motion for the particle is [1, 33]

$$\dot{\boldsymbol{x}}_k(t) = \frac{\boldsymbol{j}_k(\boldsymbol{q},t)}{|\psi(\boldsymbol{q},t)|^2}. \tag{2}$$

Here $\boldsymbol{j}$ is the probability current, defined through the quantum mechanical continuity equation $\dot{\rho} + \nabla \cdot \boldsymbol{j} = 0$, where $\rho = |\psi|^2$.

The $\boldsymbol{q}(t)$ is the actual positions of the particles at time t. According to Born's rule, the probability of finding a system in a given configuration $\boldsymbol{q}$ is given by $\rho = |\psi(\boldsymbol{q})|^2$. It can be shown that, given an ensemble of particles initially distributed in such a way, the equation of motion (2))will preserve this distribution [11]. The probabilistic interpretation of $|\psi|^2$ thus emerges as the result of repeated measurements, and not as a fundamental indeterminism.

A common critique voiced against Bohmian mechanics is that the dual role of $|\psi|^2$, as both the statistical ensemble to which the configurations belong and as a fundamental part in the equations of motion, seems unnatural and contrived. Bohm noted this, but did not regard this as a serious problem. As he relied heavily on the analogy with statistical mechanics for the probability interpretation, however, he anticipated that it might be shown that

> "an arbitrary ensemble tends to decay into an ensemble with a density of $|\psi(\boldsymbol{x})|^2$."

This would demote the role of Born's rule from a postulate to a derived principle, and it has subsequently been shown that relaxation analogous to that of statistical mechanics does occur, at least under some circumstances [33]. It has been suggested that for example particles from very distant astrophysical or cosmological sources may perhaps show a deviation from Born's rule, which might be used to test this statement experimentally [29].

4 Non-locality

An important characteristic of Bohmian mechanics is its manifest non-locality. The guiding equation of a particle depends on all other particles, given that they are entangled with the particle in question. Local deterministic theories, satisfying the Einstein locality principle, have been proven impossible since they require Bell's inequality to be fulfilled, an inequality that has been proven to be violated empirically. This does not rule out non-local deterministic theories though, since the measurement apparatus can affect not only the particle measured upon, but also distant particles. For example, in the famous EPR experiment, on whihc Bohm wrote an exposition [11], the apparatus performing the first measurement affects both the electron it measures upon and the electron that will be measured upon later by another apparatus.

The non-locality of Bohmian mechanics does, however, not imply that information can be transferred at higher velocities than the speed of light. To send information the receiver would have to know the state of the particle received if it wasn't changed by the sender's interaction with an entangled particle, and due to the relaxed nature of Bohmian particles that state can't be known.

5 The measurement problem

According to the usual interpretation of quantum mechanics, the solution to the Schrödinger equation (1), the so called wavefunction, is the most complete available description of the physical situation. The amplitude of the wave function ψ gives the probability of finding the system in a given state when measuring. According to the physicist Paul Dirac (quoted from [24])

> "[a] measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured."

It is clear, however, that there is no mechanism in the usual theoretical framework explaining why one value is measured and not another. How then to account for the distinct results of experiments? This is the measurement problem of quantum mechanics.

The most popular formulation of the measurement problem is probably the "burlesque" example of Schrödinger's cat, constructed to illustrate what Erwin Schrödinger perceived as the incompleteness of quantum mechanics [25]. Schrödinger, and later Bell [8], criticised the vague definition of what constitutes a measurement, and this has been the focal point of much debate on the philosophical interpretation of quantum mechanics. The discussion has sometimes been so heated that Bell once proposed that the term measurement should be abandoned altogether [8].

Many suggested solutions for the measurement problem have been presented. The principle of decoherence can explain why the mixed states vanish, so that no interference phenomena occur between macroscopic objects. Decoherence assures that the unfortunate cat of Schrödingers gedankenexperiment is really *either* dead *or* alive, but makes no commitment as to which of the pure states is eventually realised [8, 33]. In the view of some, the mind of the observer somehow collapses the wave function to one unambiguous value. Others have proposed that the collapse is spontaneous [7], and yet others that there is no collapse, but that the universe is constantly divided into new universes, one for each possible outcome [6]. It has also been suggested that quantum mechanics is only a tool for calculations, and that it is not to be taken as anything but a phenomenological description of the outcomes of experiments. Finally, in Bohmian mechanics, and other hidden variable theories, the measurement problem is solved by supposing that the quantum mechanical statistics arise from underlying mechanics, which are at least conceptually, if not experimentally, accessible to us. This last view is the one preferred by Bohm and Bell.

In his article of 1952 [11], Bohm suggested a solution to the measurement problem based on his reformulation of de Broglie's pilot wave theory. In the article he shows that, in this formulation, the particle and measurement apparatus have to be treated as an indivisible quantum mechanical whole in the act of measuring. This means that the experimental setup is not separate from the system being measured. He further describes how, through this interaction, a definite outcome of the experiment emerges, specified only by the initial configuration of the system being measured and of the equipment being used to perform this measurement. Since neither system nor apparatus can be arbitrarily fine tuned, however, the Bohmian formulation cannot be used to break Heisenberg's inequality, and the result of many measurements will be distributed in accordance with Born's rule, in compliance with the usual interpretation. The difference is that, in the Bohmian interpretation, the probabilistic properties of quantum mechanics, including Heisenberg's uncertainty principle, stem from

> "practical necessity and not as a manifestation of an inherent lack of complete determination in the properties of matter at the quantum level." [11]

6 Objections to Bohmian mechanics

One of the main objections towards Bohmian mechanics is that it has no counterpart to quantum field theory (QFT). QFT relies heavily on locality, and it is thus argued that such a theory might not be realizable within Bohmian mechanics. Proponents of Bohian mechanics point out that this doesn't seem to be a principal problem, it's just a matter of finding an extension of the theory to cover those results [23].

Another crucial objection is that there is no Lorentz invariant expression of the theory. Claims have even been made that non-locality and Lorentz invariance are in contradiction to each other. There is, however, other non-local theories that supposedly doesn't violate Lorentz invariance, such as the Wheeler-Feynman theory of electromagnetism, and Dürr et al. have shown in [14] that you could make Bohmian theory Lorentz invariant to some extent [23].

Deotto and Ghirardi have shown in [13] that the particle trajectories in Bohmian mechanics are undetermined, in that you can add a divergenceless vector field to the probability current and thus get a theory that gives the same statistics but where the trajectories are different [23]. Since Bohmian trajectories are supposed to be the real particle trajectories, you would have to motivate restricting the vector field to the one used in the theory. Such motivation, in terms of symmetry arguments and simplicity constraints, has been proposed by Dürr et al. in [16].

7 Summary

The de Broglie-Bohm theory is a viable alternative to ordinary quantum mechanics, in that it successfully reproduces the empirically verified statistics of quantum mechanics. That being said, it is more cumbersome to work with than the ordinary theory. In his seminal paper of 1952 [11], Bohm expressed the proposal that a consistent interpretation in terms of 'hidden' variables might open for possible extensions of quantum theory into sub-atomic regimes where it is normally not applicable. This has not been fulfilled, however, and thus the main contribution of Bohmian mechanics is perhaps to the ontological side of the field. In the words of J. S. Bell [4]

> "More importantly [...] the subjectivity of the orthodox version, the necessary reference to the 'observer,' could be eliminated."

Part II
Simulation of Bohmian mechanics

8 Introduction

As early as 1952 David Bohm anticipated that Born's rule might not be needed as postulate in Bohmian mechanics, but rather that this would be an emergent property following from the guiding equation (7). That an ensemble initially distributed according to the Bron rule is preserved by the guiding equation was shown in Bohm's original article [11]. As noted in section 3 above, he also speculated that ensembles with arbitrary distributions would converge to the $|\psi|^2$ distribution with time. This, however, is not as easy to show mathematically.

However, a sub-quantum version of the H-theorem has been proven [28]. That is, for the entropy of ρ relative to $|\psi|^2$ defined on a coarse grain, an initial decrease can be shown to take place. Furthermore, convergence has been shown to take place in numerical simulations on a particle in a two-dimensional box [30]. This report aims to produce similar numerical results but with slightly different methods and for a system not investigated in this aspect, namely a two-dimensional simple harmonic oscillator.

9 Theory

The H-function, which can be seen as the entropy of ρ relative to its supposed equilibrium $|\psi|^2$, is defined as

$$H = \int \mathrm{d}r \rho \ln \frac{\rho}{|\psi|^2} \tag{3}$$

where the integral is taken over the entire domain[1]. This exact H-function can be shown never to change if the particles of the ensemble move according to the guiding equation (7). This is, however, not true for the course-grained H-function, $\bar{H}$, defined as

$$\bar{H} = \int \mathrm{d}r \bar{\rho} \ln \frac{\bar{\rho}}{\overline{|\psi|^2}} \tag{4}$$

where $\bar{\rho}$ and $\overline{|\psi|^2}$ are ρ and $|\psi|^2$ averaged over non-overlapping coarse grain cells of size ε. The hypothesis we will try to verify is that, for a finite value of ε, $\bar{H}$ will converge towards 0, i.e., $\bar{\rho}$ will converge towards $\overline{|\psi|^2}$. This would be very similar to the convergence towards a macroscopic equilibrium in thermodynamics.

The system we will simulate is a single particle moving in a two-dimensional harmonic oscillator. The Hamiltonian for this system is given by

[1] In other contexts, the relative entropy is also referred to as as *relative information* or *Kullback information* [21].

$$\hat{\mathbf{H}} = \frac{p^2}{2m} + \frac{m\omega^2}{2}r^2 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\omega^2}{2}(x^2 + y^2) = \left(\frac{1}{2m}p_x^2 + \frac{m\omega^2}{2}x^2\right) + \left(\frac{1}{2m}p_y^2 + \frac{m\omega^2}{2}y^2\right) = \hat{\mathbf{H}}_x + \hat{\mathbf{H}}_y. \quad (5)$$

Since the Hamiltonian is clearly separable, its eigenfunctions are just a product of two eigenfunctions for a one-dimensional harmonic oscillator. An arbitrary wavefunction in this system can be written in terms of Hermite polynomials H_n, as presented in (6) [24].

$$\psi(\boldsymbol{r}, t) = \sum_{n_x, n_y} c_{n_x,n_y}\, \psi_{n_x}(x)\, \psi_{n_y}(y)\, \mathrm{e}^{\mathrm{i}(n_x+n_y+1)\omega t} \quad (6a)$$

$$\psi_n(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} (2^n n!)^{-1/2} \mathrm{e}^{-\frac{m\omega}{2\hbar}q^2}\, H_n\left(\sqrt{\frac{m\omega}{\hbar}}q\right) \quad (6b)$$

$$\sum_{n_x,n_y} |c_{n_x,n_y}|^2 = 1 \quad (6c)$$

In a single particle system, the guiding equation (2) simplifies to (7) [33].

$$\dot{\boldsymbol{r}} = \frac{1}{m}\mathrm{Im}\,\frac{\nabla\psi}{|\psi|^2} \quad (7)$$

10 The simulation

For our simulation we used a wavefunction given by c_{n_x,n_y} being zero for all modes for which $n_x > 3$ and/or $n_y > 3$. The 16 modes left (for which $n_x \leq 3$ and $n_y \leq 3$) were all given equal weight and random phases.

An ensemble of 100000 particles were assigned random positions from a Gaussian distribution of width 1.5 lengthunits. The particles are seen as being in separate but identically prepared systems, and do not interact. The same c_{n_x,n_y} are used for every particle. The units were chosen so that $\hbar$, ω, and m were all equal to 1.

If we calculate the gradient of ψ given by (6) and substitute it together with ψ itself into (7) we get a system of ODEs. For each of the particles we have solved these ODEs with given starting positions numerically to get the trajectories during a time period of 8π timeunits.

Simple Euler integration works well for areas where $|\psi|^2$ does not vanish, but as can be seen from the guiding equation (7), the velocity diverges as $|\psi|^2 \to 0$. This has the effect that particles near nodal points in ψ will be expunged, vaguely as though from centrifugal forces, due to the imperfect Euler integration [26]. Since areas where $|\psi|^2$ is small are areas where a low particle density is predicted, however, this can have the adverse effect that the relaxation is much quicker than it ought really to be, while still yielding visually perfectly sensible particle distributions. Since the goal is to study the relaxation time, the forward Euler method should be avoided, and more stable methods utilised. We opted to use *Matlabs* inherent ODE-solver `ode45`, which is based on the Runge-Kutta method.

11 Coarse graining

From the continuity equations of ρ and $|\psi|^2$ follows that the ratio $f(x,y,t) = \frac{\rho(x,y,t)}{\psi(x,y,t)^2}$ is constant along a trajectory [30]. Thus, as mentioned in section 9, relaxation does not take place in the single points. The relaxation can rather be thought of as spatial mixing of the fluctuations of $f(x,y,t)$ causing these fluctuations to take place on smaller and smaller lengthscales. In the simulations in [30] the exact values of ρ are calculated, using the fact that $f(x,y,t)$ is constant along a trajectory, and then $\bar{\rho}$ is calculated by averaging over square cells of size ε.

We just counted the number of our 100000 particles in each box to approximate the average density in the box, $\bar{\rho}$. This $\bar{\rho}$ is used for the calculation $\bar{H}$. However, for the plots we have calculated an average in overlapping cells, $\tilde{\rho}$. In $\bar{\rho}$ every value in $\bar{\rho}$ is replaced by an average of its value in the 7×7 closest cells. The purpose of this is to get smoother plots so that the similarities between $\bar{\rho}$ and $\overline{|\psi|^2}$ become more clearly visible.

It is estiamted in [30] that the relaxation time τ should depend on the coarse graining as in equation (8).

$$\tau \sim \frac{1}{\varepsilon}\frac{\hbar^2}{m^{1/2}(\Delta E)^{3/2}} = \left\{ \begin{array}{c} \hbar = m = 1, \\ \varepsilon = 10/n, \\ \Delta E = \sqrt{5/2} \end{array} \right\} = \frac{n}{10}\frac{1}{(5/2)^{3/4}} \tag{8}$$

12 Results

Two sample trajectories for particles with neighbouring starting positions are shown in figure 1. Although they follow almost the same path for a while, they eventually separate and end up far from each other. This could be a sign of chaoticity, which has been reported for Bohmian trajectories in some systems, see references in [30]. Such systems are characterised by positive *Lyapunov exponents*, which can be estimated from the simulated dynamics [12]. We have not, however, undertaken any systematic investigation to establish whether there is chaos in the system studied in this project.

In figures 2–4 the time evolutions of $\tilde{\rho}$ and $\overline{|\psi|^2}$ are presented. Already at $t = \pi/2$ the coarse-grained distribution of ρ has lost its Gaussian shape. After one period, at $t = 2\pi$, $\overline{|\psi|^2}$ has returned to its initial shape and $\tilde{\rho}$ has assumed much of the overall structure of $\overline{|\psi|^2}$. At $t = 8\pi$, the relative amplitudes of the peaks are quite consistent with those of $\overline{|\psi|^2}$.

In figure 5 the time evolution of $\overline{H}$ as defined in (4) is shown for some different coarse grains. They can be accurately fitted to exponential functions, except for some initial period time in the order of π. This divergence from exponential decrease is investigated in figure 6, where the estimated half-life τ of $\overline{H}$ as a function of the coarse graining is shown. An estimate of τ from $t \in (0, \pi/2)$ is significantly lower than that from the rest of the simulation. This can be anticipated from the appearance of the linear plot in figure 5(a), in which it is indicated that the initial decrease in $\overline{H}$ is faster than the estimate based on the whole simulation.

Also plotted is an theoretical estimation of τ adapted from [30], wherein a "rough" estimate is made on the order of magnitude of the increase of τ. The

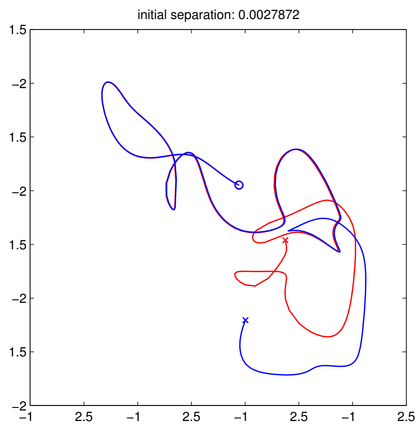


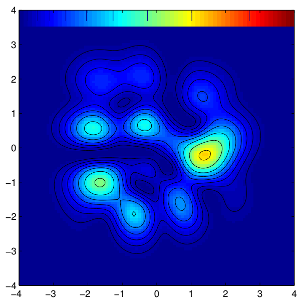
Figure 1: Two Bohmian trajectories followed from start to halfway through the simulation. Despite the starting positions (◯) being very close, the positions at $t = 4\pi$ ($\times$) are notably separated.

discrepancy between our result and the theory could be due to our simulation being done at too large time scales, which is supported by the fact that an estimate from the early part of the simulation is in better concordance with the theoretical estimate than one from the whole simulation.

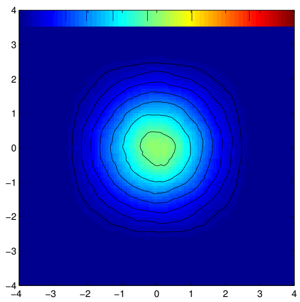
13 Discussion

The objective of this study was wanted to investigate the statement that $\rho \to |\psi|^2$ for an arbitrarily chosen initial distribution ρ, voiced in [30] and also suggested by Bohm [11]. Because of the uniqueness of the trajectories, however, in one-dimensional systems no intersections between trajectories can occur. This has the consequence that relaxation does not occur, or is very limited [30]. In the event that particles not distributed according to Born's rule are discovered, this predicts that trapping the particles in essentially one-dimensional systems may be a method for preserving this distribution. On the whole, however, two dimensions is a more interesting case for simulations. Some initial simulations for a one-dimensional harmonic oscillator were preformed, but put aside in favour of two-dimensional systems.

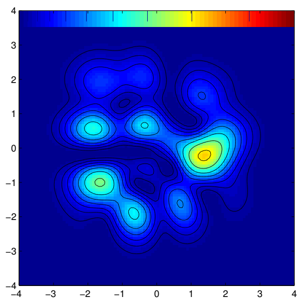
The physical situation could, for example, be an electron trapped in a lattice vacancy in an alkali halide crystal, a so called *F center*. The F in F center is an abbreviation of the German word *Farbe*, meaning *colour*. This is because defects causing such centers endow the normally transparent alkali halide crystals with colour [20]. That the resonance frequency is in the visual range can be used to make an order of magnitude estimation of the angular frequency ω of the harmonic oscillator. This yields $\omega = 2\pi\frac{c}{\lambda} \sim 10^{15}\,\mathrm{s}^{-1}$. Comparing this to the numerical results for the relaxation time, presented in figure 6, we can estimate that $\tau \sim 10^{-15}$ s, confirming that the Bohmian relaxation is a very quick process.



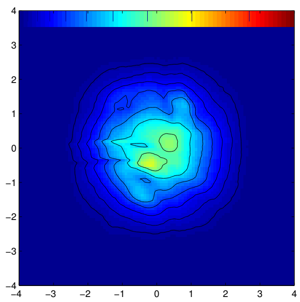
(a) $\overline{|\psi|^2}$, $t = 0$



(b) $\widetilde{\rho}$, $t = 0$

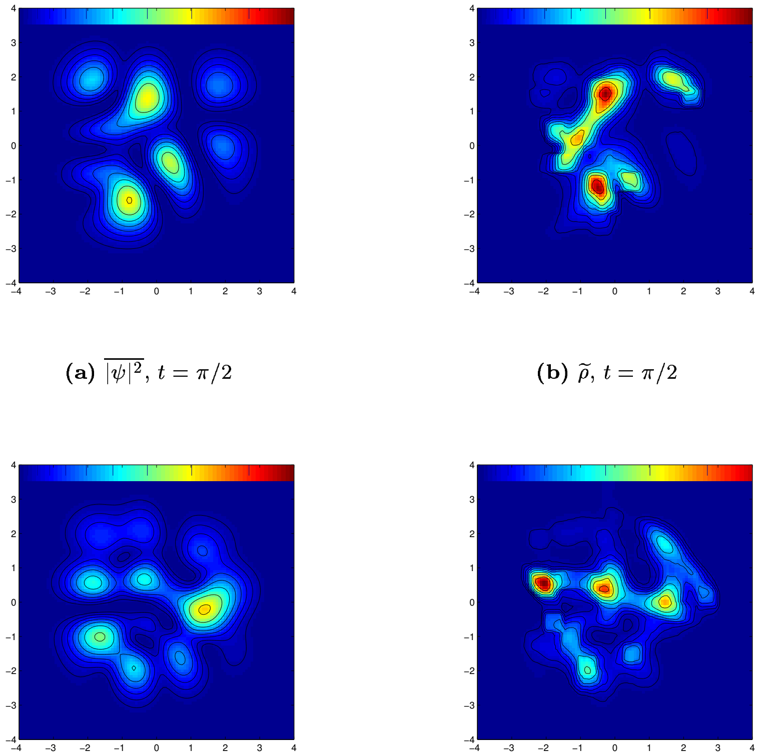


(c) $\overline{|\psi|^2}$, $t = 0.008\pi$



(d) $\widetilde{\rho}$, $t = 0.008\pi$

Figure 2: Comparison of $\overline{|\psi|^2}$ and $\widetilde{\rho}$ before and after the very first time step



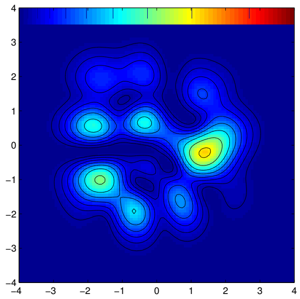
(a) $\overline{|\psi|^2}$, $t = \pi/2$

(b) $\widetilde{\rho}$, $t = \pi/2$

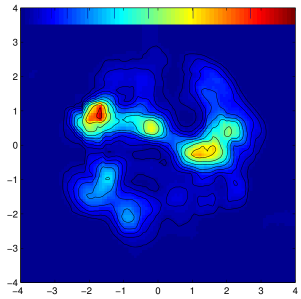
(c) $\overline{|\psi|^2}$, $t = 2\pi$

(d) $\widetilde{\rho}$, $t = 2\pi$

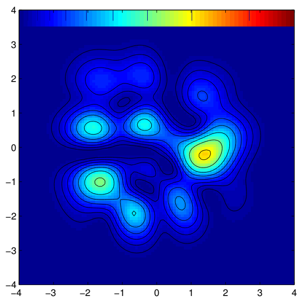
Figure 3: Comparison of $\overline{|\psi|^2}$ and $\widetilde{\rho}$ at times $t = \pi/2$ and $t = 2\pi$, in the first half of the simulation



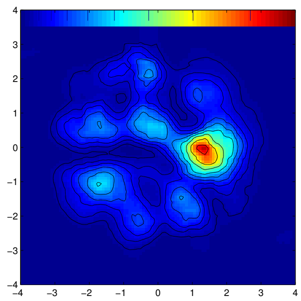
(a) $\overline{|\psi|^2}$, $t = 4\pi$



(b) $\widetilde{\rho}$, $t = 4\pi$

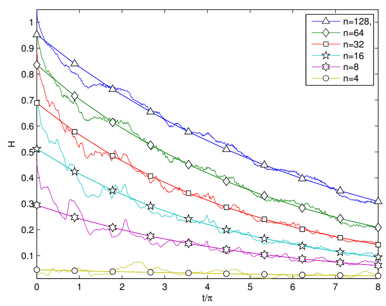


(c) $\overline{|\psi|^2}$, $t = 8\pi$

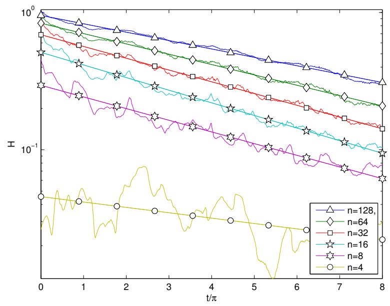


(d) $\widetilde{\rho}$, $t = 8\pi$

Figure 4: Comparison of $\overline{|\psi|^2}$ and $\widetilde{\rho}$ after half the simulation, $t = 4\pi$ time and after the final time step, $t = 8\pi$



(a)



(b)

Figure 5: Relative entropy $\overline{H}$ on some different $n \times n$ coarse grains together with fitted exponential functions

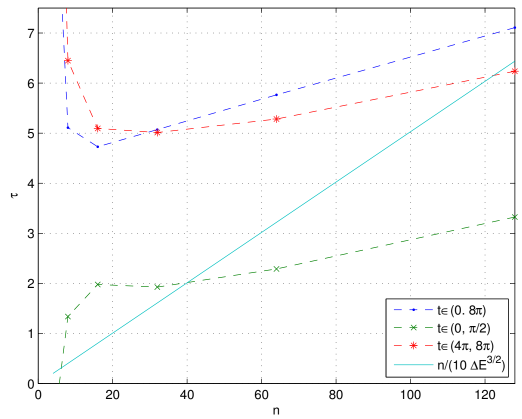


Figure 6: Half-life τ in units of π as a function of n. One cycle for the oscillator is 2π.

14 Conclusions

From the results, as presented in figures 2–4, we can conclude quantitatively that relaxation does occur in our system. The convergence $\overline{\rho} \to \overline{|\psi|^2}$, as measured by the relative entropy $\overline{H}$, is observed to be exponential in time, as predicted [30]. This can be seen in figure 5.

The halflife τ of the relaxation was estimated from the generated data, ans is presented in figure 6. It was found that τ is of the right order of magnitude. A comparison to a physical system showed that the relaxation is of the order of femtoseconds, which might motivatate why relaxation is not observed on a macroscopic level, and why ensembles not obeying Born's rule are not observed. The dependence of τ on the coarse graining (equation (8)) could not be confirmed. We believe that this may be due to lack of data for the very onset of the relaxation, and more detailed studies of this regime may be fruitfull.

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